Word Problems 6: The Mixed-Rate Problems #3

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Abstract

In this algebra word problem note, we use the Scheme to solve our third attempt at what I refer to as a 'mixed-rate problem'. In this type of problem, two or more 'machines' work together at generally different rates to produce subtotals that add to a total. The added twist to these problems involves working with direct and inverse relations among variables.

1 Introduction

This time our 'machines' are first two copywriters working on the same copyreading job; and the second are two different kinds of text (at two different point sizes) vying for space on the same page. Sound interesting? We'll save that problem for the last, and first work the problem of two copywriters cooperating on a project together .

2 Word Problem #6.1

Martin and Wood are hired to do a copyreading job together. Working alone, Martin could do 2/3 rds of the job in 15 days. And Wood, working alone, could do the job in 9 days. If they work together (start to finish) how long will it take?

3 Solution 6.1.1: Conceptualizing the Problem

As usual,

$$1 \text{ job} = (\text{part of job done by Martin}) + (\text{part of job done by Wood})\,. \tag{1}$$

For our next refinement, we use the average rate R at which each copywriter completes the job and then multiply by the time he takes on the job.

$$1 = R_M T_M + R_W T_W\,, \tag{2}$$

where R_M can be determined by a proportion[1] (below), $T_M = T_W \equiv T$, and $R_W = \frac{1}{9}$ job/day.

$$\frac{\frac{2}{3}\text{job}}{15 \text{ days}} = \frac{1[\text{job}]}{x} \tag{3}$$

Solving for x, we get $x = \frac{45}{2}$ days. Therefore, $R_M = \frac{2}{45}$ job/day. Substituting all this into (2), we get

$$1 = (\frac{2}{45} + \frac{1}{9})T\,, \tag{4}$$

Solving this for T, we get $T = \frac{45}{7}$ days. Or, 6 days 10 hours 17 minutes 8.571 seconds.[2] If you think about it, this result is strange. If Martin and Wood put in 8-hour work days, then how do we interpret the 10 hours? Maybe it's better just to roundup to 7 days to complete the job. This is a clear example of how imprecise given information on the problem can lead to an ambiguous result.

4 Word Problem #6.2

Mona, the editor of a textbook at Labrador Scientific Publishers, has had a heated argument with an author for the printing of the 2nd edition of his book. The author insisted on adding words to the text and to a footnote on the last page. As a result, Mona found that the footnote no longer fit entirely on the 'last' page and instead was partly at the bottom of the next page – an otherwise blank page. After another heated discussion with the author late in the evening before the book was to go to print the next morning, she was given permission to alter the point sizes of the text on the last page to make the footnote fit on the page it's supposed to be on.

Not wanting to wait until the next day to solve this problem, Mona decided to try an algebraic solution before going to sleep. Normally, a page with only regular text needs 200 word at 10 point size to fill it up.[3] The page has 185 regular text words at 10 points. To make more room on the page, she has chosen to change that point size to 9.5 points. She also has 35 words in the footnote at 9 points, which is too much. She has the choices of 8, 7, or 6 point sizes to choose form, wanting the largest of the three that will just fit the footnote on the page.

5 Solution 6.2.1: Conceptualizing the Problem

So, we have the contributions to the whole page of (regular test at 9.5 points) and (the footnote text at a smaller point size), and their sum, to fit on one page, must be less than or equal to 1 page.

[1]Searching the problem for a proportion was listed as one of the main searching points in the list of the Scheme procedures.

[2]I thank wolframalpha.com for assisting in this solution.

[3]I've merely made up convenient numbers for easier computation.

Therefore,

$$\begin{bmatrix}\text{part of page taken by}\\ \text{9.5 point regular text}\end{bmatrix} + \begin{bmatrix}\text{part of page taken by}\\ \text{footnote at smaller size}\end{bmatrix} \leq 1 \text{ page}\,. \tag{5}$$

I promised the reader some notes ago that we would encounter inequalities and here we found one. Let's represent this problem in what will soon become an all too familiar graphic relating 1) parts to wholes and 2) 'before states' to 'after states'.

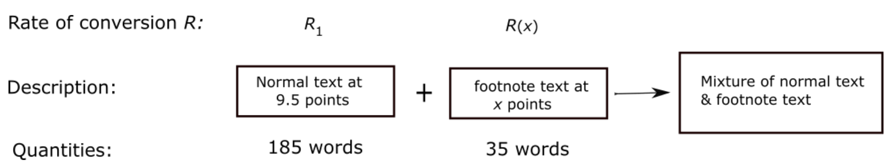

Figure 1. This graphic represents our division of text into a contribution at 9.5 points and another at x points (to be determined). The R's are the conversion factors ('machines') that change words at a given point size to area on a page.

6 Solution 6.2.2: Solving the Problem

Now, we're going to make a simple assumption that we cannot prove, but it makes sense for an approximation: For any point size, there is an inverse relation between point size and the number of words. Put another way, if we decrease the point size a bit, we will increase the number of words that can fit on a page a bit. Therefore, we have (for text at 9.5 points) that

$$200 \text{ words} \times 10 \text{ points} = w \text{ words} \times 9.5 \text{ points}\,. \tag{6}$$

Solving this for w, we get

$$w = \frac{200 \times 10 \text{ points}}{9.5 \text{ points}} \approx 210.5\,. \tag{7}$$

What this means is that if all the text were at 9.5 points, a full page would take 210.5 words. Now we make another unprovable, but reasonable, assumption that for any fixed point size, the number of words is directly proportional to the amount of page taken up. In this case, we get

$$\frac{210.5 \text{ words}}{1 \text{ page}} = \frac{185 \text{ words}}{y}\,. \tag{8}$$

Therefore, $y = 0.879$ part of the final page. Therefore, by similar computation, we need to find the largest point size, out of 8,7,6 points, to apply to the 35 words in the footnote and have the result take up $\leq 1 - 0.879 = 0.121$ part of a page.

Before moving on, I want to relate the y value above to the R_1 in Figure 1. The y value is the subtotal of the amount of page consumed by words at 9.5 points. As an equation we could write

$$y = R_1 Q_1 = \left(\frac{1}{210.5}\,\frac{\text{page}}{\cancel{\text{word}}}\right)(185\ \cancel{\text{words}}) = 0.879\ \text{page}\,. \tag{9}$$

This is, in essence, what we did to get y from the proportion (8). Equation (9) merely formalizes the calculation into the product of a rate R_1 times a quantity Q_1 $(= 185)$, which is what these subtotals almost always are.

In the calculations below, I continue to use y for subtotals of a page consumed by text at the various point sizes x, which are related to the conversion factors $R(x)$ and quantities $Q(x) = 35$ (a constant in this problem).

Let's begin at 8 points: The analogous equation to (6) is

$$200\ \text{words} \times 10\ \text{points} = w\ \text{words} \times 8\ \text{points}\,. \tag{10}$$

Solving this for w, we get

$$w = 250\,. \tag{11}$$

Thus, a full page at 8 points requires 250 words. By an equation similar to (8), we get

$$\frac{250\ \text{words}}{1\ \text{page}} = \frac{35\ \text{words}}{y}\,, \tag{12}$$

and $y = 0.14$. But this goes over the 0.121 part of a page we are allowed to use for the footnote.

So, let's go down to 7 points: Again the analogous equation to (6) is

$$200\ \text{words} \times 10\ \text{points} = w\ \text{words} \times 7\ \text{points}\,. \tag{13}$$

Solving this for w, we get

$$w = 285.7\,. \tag{14}$$

Thus, a full page at 7 points requires 285.7 words. By an equation similar to (8), we get

$$\frac{285.7\ \text{words}}{1\ \text{page}} = \frac{35\ \text{words}}{y}\,, \tag{15}$$

and $y = 0.123$. But this just goes over the 0.121 part of a page we are allowed to use for the footnote.

So, let's go down to 6 points: The analogous equation to (6) is

$$200\ \text{words} \times 10\ \text{points} = w\ \text{words} \times 6\ \text{points}\,. \tag{16}$$

Solving this for w, we get

$$w = 333.3\,. \tag{17}$$

Thus, a full page at 7 points requires 333.3 words. By an equation similar to (8), we get

$$\frac{333.3 \text{ words}}{1 \text{ page}} = \frac{35 \text{ words}}{y}\,, \tag{18}$$

and $y = 0.105$, which is less than or equal to the 0.121 part of a page we are allowed to use for the footnote. And we finally found our correct point size. Updating (5), we get

$$\begin{bmatrix}\text{0.879 part of page taken}\\ \text{by 185 words at 9.5 pts}\end{bmatrix} + \begin{bmatrix}\text{0.105 part of page taken}\\ \text{by 35 words at 6 pts}\end{bmatrix} \leq 1 \text{ page}\,. \tag{19}$$

When Mona gets to work the next morning, she will tell the printers her calculated results for point sizes and then wait to see if those values actually work in practice.

7 Conclusion

The problem of Martin and Wood is the standard fare analysis with only one direct proportion to include in the calculations.

The problem of solving for the best point size is not simple, is it? It has both direct and inverse relations to deal with. It has multiple parts based on a discrete parameter, requiring an exhaustive search technique.

As a final comment on both of these problems, I want to remark that their greatest virtue, at least to me, lies in their real-world appeal. If you recall the 'coins in a jar' problem, that problem had the feel of being made-up and unrealistic. But it's easy to imagine the problems in this note coming right out of real-world situations. They are the answer to the high-schooler's perennial question, "When will I ever use this stuff?"[4]

[4]The answer to this question is simple: When you make up algebra word problems.