Cardinality of the Power Set

P. Reany

February 2, 2020

Abstract

We are interested in answering this question: Is the cardinality of the power set always greater than the cardinality of the set it came from?

1 Theorem

Comment: The beautiful and clever proof presented herein is not new, though its presentation in the form of a flowchart might be.

Let M be a non-empty set. Let $P(M)$ be the *power set* of M, that is, the set containing all the subsets of M. We will represent the cardinality of a set by vertical bars. Thus, the cardinality of set M, say, is $|\, M \,|$.

The statement of the theorem is this: For all non-empty sets M

$$|\, P(M) \,| > |\, M \,| \, . \tag{1}$$

The theorem is obvious when M is finite, but subtle things can happen when M is infinite, so we need a careful proof in that case. The simplest way to achieve this goal is to craft a proof that never relies on M being finite.

We have three mutually exclusive possibilities:[1] Either,

Case 1: $|\, P(M) \,| < |\, M \,|$,

Case 2: $|\, P(M) \,| = |\, M \,|$,

Case 3: $|\, P(M) \,| > |\, M \,|$.

So, to prove that Case 3 is true, we need only falsify Cases 1 and 2.

For starters, Case 1 is obviously false, whether M is finite or infinite. That leaves us with Case 2. If Case 2 were true, then it would be possible to find a bijection φ from M to $P(M)$. If we can show that such a bijection is not possible, then only Case 3 remains, and we're finished.

Our strategy is simple: We assume the existence of φ and then derive a contradiction. The proof is contained in the flowchart depicted in Figure 1. It

[1] This claim is derived from the *Law of the Trichotomy of the Cardinals*, which is analgous to the familiar case when comparing magnitudes of real numbers.

introduces the notion of a *humble* element of the original set M, which is purely heuristic.

First, I arbitrarily interpret φ as the map that takes in an element, say a, of M and names the subset of elements of M that a 'talks favorably about'. Now, a may choose to talk favorably about itself, or not. In the latter case, $a \notin \varphi(a)$, in which case it is referred to as a "humble" element. But in the former case, $a \in \varphi(a)$, in which case it is referred to as "not humble." On to the flowchart:

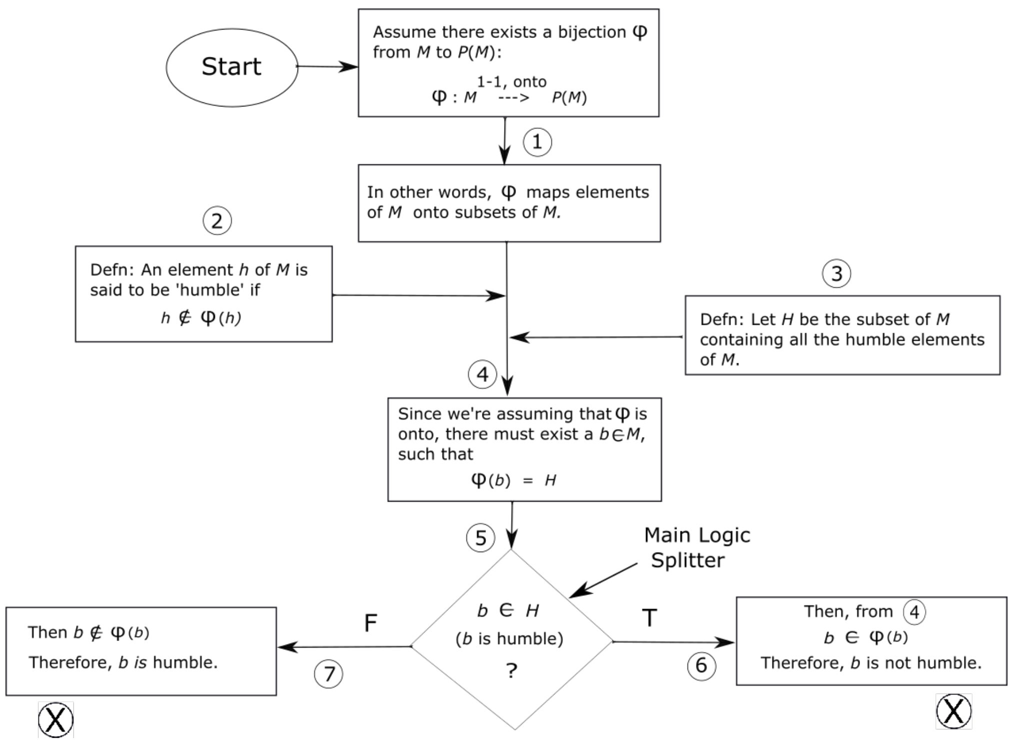

Figure 1. This flowchart shows that assuming the existence of a bijection φ from M to $P(M)$ creates a contradiction, falsifying the assumption that φ exists.

QED ■