More proofs in plane geometry using vector methods

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, April 1991. We employ the same techniques as last time, though on two new problems.

1 Problem 1.

Show that in a right triangle the midpoint of the hypotenuses is equidistant from the vertices.

We begin by drawing a figure such as Figure 1. The right angle is at vertex C. Since the parts of the figure are already simple polygons we won't have to subtract anything until we arrive at the regions, translating their information into circuit/loop equations as is done in Table 1.

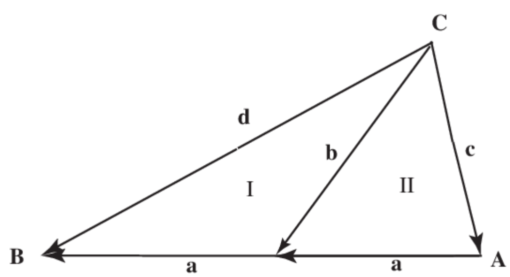

Figure 1

Region	Loop Equation
I	$\mathbf{d} = \mathbf{b} + \mathbf{a}$
II	$\mathbf{c} = \mathbf{d} - \mathbf{a}$

Table 1: Circuit equations for Figure 1.

I need to show that $a = b$, where $a \equiv |\mathbf{a}|$, etc. To do this, I will use the standard trick of substituting the circuit equations into the main structure equation

$$\mathbf{d} \cdot \mathbf{c} = 0\,. \tag{1}$$

Thus,

$$\mathbf{d} \cdot \mathbf{c} = (\mathbf{b} + \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = |\mathbf{b}|^2 - |\mathbf{a}|^2 = b^2 - a^2 = 0\,. \tag{2}$$

Thus $b = a$, as I needed to show.

2 Problem 2

Show that the perpendicular bisectors of a triangle are concurrent at a point. In the figure below, vectors **e** and **d** are vectors along the perpendicular bisectors of their respective sides.

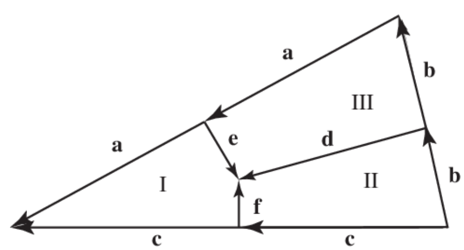

Figure 2

The trick is to recognized that two of the perpendicular bisectors will of necessity meet at a point; then we must show that the vector from that intersection point to the midpoint of the third side is perpendicular to that third side. Stating this in terms of vector equations, we choose to set the two structure constraints

$$\mathbf{e} \cdot \mathbf{a} = 0\,, \tag{3}$$

$$\mathbf{d} \cdot \mathbf{b} = 0\,, \tag{4}$$

in the hope of establishing that

$$\mathbf{f} \cdot \mathbf{c} = 0\,. \tag{5}$$

Alternatively, we can start with the expression $\mathbf{f} \cdot \mathbf{c}$ and substitute loop-equation constraints into it, and, with the two constraints (3) and (4) arrive at zero. Let's use the latter method.

Region	Loop Equation
I	$\mathbf{f} = \mathbf{c} - \mathbf{a} + \mathbf{e}$
II	$\mathbf{f} = -\mathbf{c} + \mathbf{b} + \mathbf{d}$
III	$0 = \mathbf{b} + \mathbf{a} + \mathbf{e} - \mathbf{d}$

Table 2: Circuit equations for Figure 2.

The problem for us is to determine the best choice for $\mathbf{c}$ and $\mathbf{f}$ in terms of the loop equations presented in Table 2. We can use the following simplified expressions for $\mathbf{f}$ and $\mathbf{c}$ by taking linear combinations of the loop equations:

$$\text{Adding I+II+III yields:} \quad 2\mathbf{f} = 2\mathbf{b} + 2\mathbf{e} \quad \Rightarrow \quad \mathbf{f} = \mathbf{b} + \mathbf{e}\,, \tag{6}$$

$$\text{Taking I}-\text{II}-\text{III yields:} \quad 0 = 2\mathbf{c} - 2\mathbf{b} - 2\mathbf{a} \quad \Rightarrow \quad \mathbf{c} = \mathbf{a} + \mathbf{b}\,. \tag{7}$$

We can now substitute these expressions into the LHS of the ShowThat equation (5):

$$\mathbf{c} \cdot \mathbf{f} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{e}) = \mathbf{a} \cdot (\mathbf{b} + \mathbf{e}) + \mathbf{b} \cdot (\mathbf{b} + \mathbf{e})\,. \tag{8}$$

We can use Eq. (4) for some assistance here:

$$\mathbf{d} \cdot \mathbf{b} = (\mathbf{b} + \mathbf{a} + \mathbf{e}) \cdot \mathbf{b} = 0\,, \tag{9}$$

where we used the loop equation from Region III. Solving this we get

$$(\mathbf{b} + \mathbf{e}) \cdot \mathbf{b} = -\mathbf{a} \cdot \mathbf{b}\,. \tag{10}$$

Using this result in (8), we get

$$\mathbf{c} \cdot \mathbf{f} = \mathbf{a} \cdot (\mathbf{b} + \mathbf{e}) - \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{e} = 0\,, \tag{11}$$

where we finally used constraint (3). And we are finished.