Proofs in plane geometry using vector methods

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Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, January, 1991.

In this article I want to demonstrate some heuristics for solving geometric proofs using the methods of vector algebra. An understanding of vector algebra is assumed. My plan is to present three problems solved in full, and to aid the reader in finding the heuristic principles involved.

It's very important in solving these problems to understand what it means to use the method of vectors: We will be *translating* the geometric information in the problem into operations on vectors, such as vector lengths or the angle between two vectors, and doing so by use of vector dot and/or cross products on vectors. We will call these relationships *structure* conditions or equations, because they help determine the geometric structure of the figure.

1 Problem 1:

Using the methods of vector algebra show that an angle inscribed in a semicircle is a right angle.

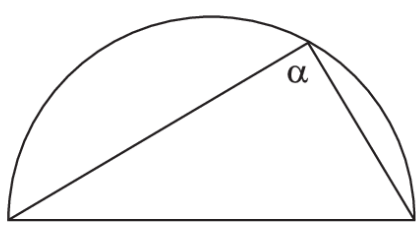

Figure 1.1

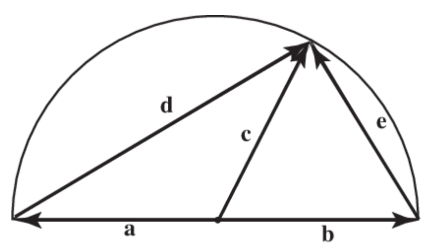

Figure 1.2

Step 1. We start off by drawing a figure of the given problem, as in Figure 1.1.

Step 2. Next we label the parts of the figure and when possiblerestate all other given information into vector form, as in Figure 1.2. For the problem here, to prove that α is a right angle, it is sufficient to show that

$$\mathbf{d} \cdot \mathbf{e} = 0\,. \tag{1}$$

You may wonder what motivated me to draw in and label the vector **c**, but that's really easy to see after I explain the heuristic technique I call *The Method Of Subtraction* in Step 4.

Step 3. This step reduces the number of independent vector labels by solving as many vectors as possible in terms of as few vectors as possible. There is no unique way to do this generally, so choose the ones you want to remain, and solve for the others in terms of them. In this case I'll choose to write **b** as $-\mathbf{a}$, and keep **c**. Then, vectors **d** and **e** can be written in terms of vectors **a** and **c**. The result is displayed in Figure 1.3.

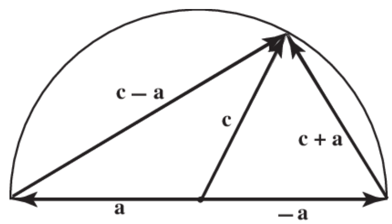

Figure 1.3

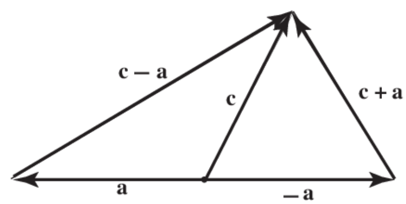

Figure 1.4

Step 4. OK, it's time to tell you the secret to The Method of Subtraction: One at a time, subtract from the figure any convenient part, while translating its geometric *information* into algebraic form. It's obvious that the semicircle should be the first thing to go! Why? Simply because vector methods are tailor-made to use with line segments, not with circles *per se*. So to remove the semicircle while leaving the information that all three vertices are the same distance from the center of the circle, I added the directed line segment **c** to the representation in Figure 1.2. Thus, the truncated figure in Figure 1.4 requires the additional added algebraic information that $|\,\mathbf{a}\,|^2 = |-\mathbf{a}\,|^2 = |\,\mathbf{c}\,|^2 = r^2$, where r is the radius of the circle. We can write this more simply as

$$\mathbf{a}^2 = \mathbf{c}^2 = r^2\,. \tag{2}$$

The only parts left should be simple polygons, which we can translate by the method given in the next step.

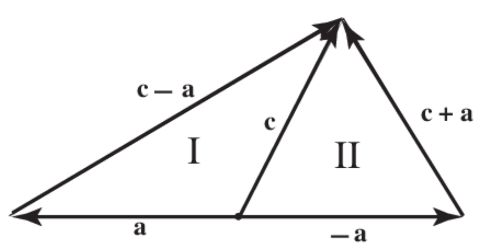

Figure 1.5

Step 5. This step is used to translate the remaining information contained in the simple polygons. For each simple polygon, which we will refer to as a *region*, we will label it with a roman numeral as in Figure 1.5. Then we will write down for each region its vector *circuit* or *loop* equation, which is a vector equation that adds up all the vectors forming the perimeter, which adds to zero. Or another scheme is to solve for one side of a simple polygon in terms of the sum of all the other sides (the vectors being added either in a clockwise or counterclockwise manner). Admittedly, this particular problem does not reveal the power of this method, but don't worry, the next two will.

Step 6. We now have enough information to solve the problem algebraically. Harkening back to (1), we see we need to show that

$$(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a}) = \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} = r^2 - r^2 = 0 \tag{3}$$

where we have used that $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$, confirming Eq. (1); hence $\alpha = 90$. And the proof is finished.

Before going on to the next problem, I want to point out that all the vector information is neatly divisible into two classes: The class of circuit equations, which contains topological information, and the class of structure equations, which contain everything else (i.e., the metrical information). The latter class contains everything involving dot products, cross products, and vector lengths.

I have a couple remarks on Problem 1 and its solution in retrospect: First, there is a third Region (III) I didn't label, namely, the region composed of the sum of Regions I and II, that is, the big triangle. The information contained in these regions is not independent of each other, and so the problem solver has to make a choice of which regions to use and which to leave out — formally, at least. In the next problem there will be many regions to choose from, and the choosing may be the most difficult part of the solution.

My second remark is that I wish I had kept vectors $\mathbf{d}$ and $\mathbf{e}$ and just stated the obvious loop equations:

Region	Loop Equation
I	$\mathbf{d} = -\mathbf{a} + \mathbf{c}$
II	$\mathbf{e} = -\mathbf{b} + \mathbf{c} = \mathbf{a} + \mathbf{c}$

Table 1: Note: $\mathbf{b} = -\mathbf{a}$.

2 Problem 2:

Show that the median of one vertex V of a triangle is bisected by a line segment from either one of the other vertices if that segment divides the side it intersects into segments whose lengths ratio is 1 : 2, the smaller segment being adjacent to vertex V.

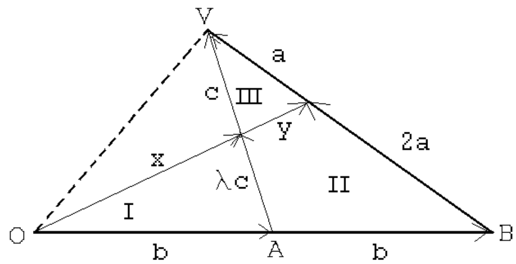

Figure 2.1

Step 1. We begin with Figure 2.1, displayed above. The line segment AV is the median at vertex V because it bisects the opposite side OB, hitting it at the midpoint A. The vector $\mathbf{y}$ starts at point O and stops at side VB, dividing it into segments of ratio 1 : 2, the shorter side being closest to V. The vector $\mathbf{x}$ starts at vertex O and continues along vector $\mathbf{y}$ until it hits line segment VA, dividing it into two segments $\mathbf{c}$ and $\lambda\mathbf{c}$. Since we are to show that this procedure has bisected VA, we must show that $\lambda = 1$.

On careful examination of Figure 2.1, we see that there are exactly two pieces of structure information not already nailed down.[1] First, that $\mathbf{x} \times \mathbf{y} = 0$ (ensuring that $\mathbf{x}$ and $\mathbf{y}$ lie on the same line).[2] And second, the value of λ. Logically, resolving the former should lead us to the latter; hence, our Main Structure Equation is

$$\mathbf{x} \times \mathbf{y} = 0\,. \tag{4}$$

I'm now adding into the mix two regions not manifest in the figure: First, Region IV, which is the triangular region being the sum of regions II and III. Second, Region V, which is the triangular region being the sum of regions I and II.

This brings us to:

Step 2. Write down the five loop equations:

$$\text{(Region I)} \quad \mathbf{x} = \mathbf{b} + \lambda\mathbf{c} \tag{5}$$

$$\text{(Region II)} \quad \mathbf{y} = -\lambda\mathbf{c} + \mathbf{b} + 2\mathbf{a} \tag{6}$$

$$\text{(Region III)} \quad \mathbf{y} = -\mathbf{c} - \mathbf{a} \tag{7}$$

$$\text{(Region IV)} \quad 3\mathbf{a} = -\mathbf{b} + (\lambda + 1)\mathbf{c} \tag{8}$$

$$\text{(Region V)} \quad \mathbf{y} = 2\mathbf{b} + 2\mathbf{a} \tag{9}$$

Now, substitute $\mathbf{x}$ in the main structure equation (4) its value from Eq. (5), and

[1] The dotted segment OV is a nonplayer, since it will be determined as soon as triangles I, III, and quadrilateral II are determined.

[2] It's also possible to use as an alternative for this condition that $\mathbf{x} = t\mathbf{y}$, where t is a real number. I haven't tried this approach, but why introduce another variable?

substitute for $\mathbf{y}$ its value from (9):

$$(\mathbf{b} + \lambda\mathbf{c}) \times (2\mathbf{b} + 2\mathbf{a}) = 0\,. \tag{10}$$

On expanding this and then simplifying, we get

$$(-\mathbf{b}) \times (\lambda\mathbf{c} - \mathbf{a}) + \lambda\mathbf{c} \times \mathbf{a} = 0\,. \tag{11}$$

Now, solving for $-\mathbf{b}$ in Eq. (8) and substituting into this last equation, we get

$$3\lambda\mathbf{a} \times \mathbf{c} + (\lambda + 1)\mathbf{c} \times \mathbf{a} + \lambda\mathbf{c} \times \mathbf{a} = 0\,, \tag{12}$$

and since $\mathbf{c} \times \mathbf{a}$ $(= -\mathbf{a} \times \mathbf{c})$ is not zero, we require that $\lambda = 1$, as required.

3 Problem 3:

Show that the altitudes of a triangle are concurrent.

Step 1. We know that the altitudes meet in pairs, so we know that the points labeled $\mathbf{x}, \mathbf{y}, \mathbf{z}$ exists as demonstrated in Figure 3.1.

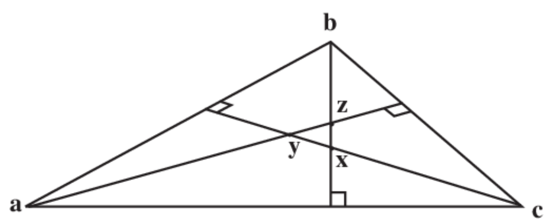

Figure 3.1

This time, however, I want to try a different labeling scheme by using the difference of points to label the line segments. For instance, $\mathbf{b} - \mathbf{z}$ is the vector from $\mathbf{z}$ to $\mathbf{b}$. This brings us to Figure 3.2.

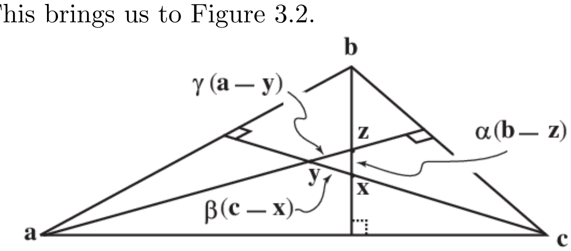

Figure 3.2

That there are three altitudes (line segments intersecting in pairs to form right angles), the structure information takes the form:

$$(\mathbf{b}-\mathbf{z})\cdot(\mathbf{a}-\mathbf{c})=0\,, \tag{13}$$

$$(\mathbf{a}-\mathbf{y})\cdot(\mathbf{b}-\mathbf{c})=0\,, \tag{14}$$

$$(\mathbf{c}-\mathbf{x})\cdot(\mathbf{b}-\mathbf{a})=0\,. \tag{15}$$

It is sufficient for solving our problem to show that $\alpha=\beta=\gamma=0$. And I'm going to use a standard cheat: I'm going to show that $\alpha=0$ and claim that by the symmetry by which the variables enter this problem, that the exact same method will show that β and γ are also zero. Now, we won't need all the possible circuit (loop) equations here, so a judicious choice will save the day.

Remark: It's reasonable to assume from the outset that all three structure equations are necessary for this proof; hence, it's reasonable to expect that we must incorporate each of them in the solution somewhere.

We start with the circuit equation for triangle **bxc**.

$$(1+\alpha)(\mathbf{b}-\mathbf{z})=(\mathbf{c}-\mathbf{x})+(\mathbf{b}-\mathbf{c})\,. \tag{16}$$

Notice that our structure equations are scalar equations and our loop equations are vector equations; thus, to put them on an equal footing, we must either dot the loop equations to scalarize them or else substitute the loop equations into the structure equations. In the process of this proof, we'll do both. By dotting (16) through by $(\mathbf{b}-\mathbf{a})$, we can invoke constraint (15):

$$(1+\alpha)(\mathbf{b}-\mathbf{z})\cdot(\mathbf{b}-\mathbf{a})=(\mathbf{c}-\mathbf{x})\cdot(\mathbf{b}-\mathbf{a})+(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})\,, \tag{17}$$

where the first term on the RHS drops out, yielding

$$(1+\alpha)(\mathbf{b}-\mathbf{z})\cdot(\mathbf{b}-\mathbf{a})=(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})\,. \tag{18}$$

We have two constraints left to use. Let's look at constraint (14). We need another circuit equation to use that. From triangle **azb**, we get that

$$(1+\gamma)(\mathbf{y}-\mathbf{a})+(\mathbf{b}-\mathbf{z})=(\mathbf{b}-\mathbf{a})\,. \tag{19}$$

By dotting this through by $(\mathbf{b}-\mathbf{c})$ and using (14), we get

$$(\mathbf{b}-\mathbf{z})\cdot(\mathbf{b}-\mathbf{c})=(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})\,. \tag{20}$$

Now we have a clever way to employ the last constraint, Eq. (13). Since $\mathbf{b}-\mathbf{c}=(\mathbf{b}-\mathbf{a})+(\mathbf{a}-\mathbf{c})$, then substituting this into the LHS of the last equation yields

$$(\mathbf{b}-\mathbf{z})\cdot[(\mathbf{b}-\mathbf{a})+(\mathbf{a}-\mathbf{c})]=(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})\,, \tag{21}$$

which by (13) becomes

$$(\mathbf{b}-\mathbf{z})\cdot(\mathbf{b}-\mathbf{a})=(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})\,. \tag{22}$$

Now, using this in the LHS of (18), yields

$$(1+\alpha)(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})=(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})\,, \tag{23}$$

which requires that $\alpha=0$. Similarly, $\beta=\gamma=0$, and we are done.