Notes On Vieta's Formulas

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Abstract

We wish to characterized how the roots of polynomials depend on their coefficients. Cyclotomic polynomials.

1 Introduction

We write as our 'general' quadratic equation as

$$x^2 + bx + c = 0\,, \tag{1}$$

where b and c are complex numbers. Believing that every quadratic equation has two roots x_1 and x_2, so that the quadratic can be factored as

$$(x - x_1)(x - x_2) = 0\,, \tag{2}$$

we can find the values of those roots in terms of the coefficients of the equation. How? First, we expand (2) to get

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0\,. \tag{3}$$

The polynomials $x_1 + x_2$ and x_1x_2 are said to be *symmetric* because, under any permutation of subscripts on the x's, the polynomials are unchanged.

Now we equate like coefficients between (1) and (3), to get

$$x_1 + x_2 = -b\,, \tag{4}$$

$$x_1x_2 = c\,. \tag{5}$$

If we want to look specifically at

$$x^2 = 1\,, \tag{6}$$

we can set $b = 0$ and $c = -1$, to get

$$x_1 + x_2 = 0\,, \tag{7}$$

$$x_1x_2 = -1\,, \tag{8}$$

which has solutions $x = \pm 1$.

From the fundamental theorem of algebra, we know that every nth degree polynomial., has n roots in the complex plane. Therefore the equation

$$x^n = 1\,, \tag{9}$$

has n roots, each of which is unity when taken to the nth power. This is possible by choosing the complex number $\alpha = e^{2\pi i/n}$ to represent a primitive root of (9), with factorization

$$(x - \alpha^0)(x - \alpha^1) \cdots (x - \alpha^{n-1}) = 0\,, \tag{10}$$

For $n = 3$, we get

$$(x-1)(x-\alpha)\;(x-\alpha^2) = 0\,, \tag{11}$$

which expands to

$$x^3 - (\alpha^2+\alpha+1)x^2 + (\alpha^2+\alpha+1)x - 1 = x^3 - 1\,. \tag{12}$$

For these two polynomials to be equal, it had better be true that

$$\alpha^2 + \alpha + 1 = 0\,. \tag{13}$$

We'll prove this as a corollary of the general case that, for $\alpha = e^{2\pi i/n}$ for $n \geq 2$:

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = 0\,. \tag{14}$$

Define

$$S = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}\,. \tag{15}$$

Then

$$\alpha S = \alpha + \alpha^2 + \cdots + \alpha^{n-1} + \alpha^n\,. \tag{16}$$

Subtracting (16) from (15), yields

$$S - \alpha S = 1 - \alpha^n\,. \tag{17}$$

And then solving this for S yields

$$S = \frac{1-\alpha^n}{1-\alpha}\,. \tag{18}$$

But since α is a root of (9), then $1 - \alpha^n = 0$ therefore $S = 0$. What this means is that the sum of all the roots to (9) is zero.